STATIONARY FLOW OF AN INCOMPRESSIBLE CONDUCTING FLUID PAST A CURRENT-CARRYING CYLINDRICAL BODY

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§ 1. STATEMENT OF THE PROBLEM

This problem may be broadly stated thus: a homogeneous stream of a conducting fluid flows past a cylindrical dieletric body containing a number of straight conductors parallel to the cylinder axis through which flows electric current generating a magnetic field. The velocity vector of the stream lies in a plane normal to the cylinder axis. The velocity field of the stream and the magnetic field distribution in and around the cylinder are to be determined.

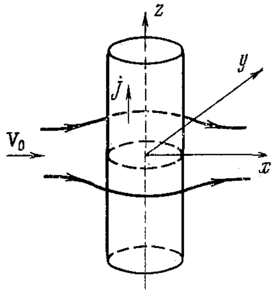

Fig. 1 Flow diagram

It is assumed in the following that the Hall effect, and the effects associated with viscosity and thermal conductivity can be neglected, and that the fluid conductivity σ and its density ρ are constant. We further assume that all magnitudes depend on x and y only (the chosen coordinate system is shown in Fig. 1), i.e., the problem is two-dimensional, and there are no external electric field sources (E = 0). With these assumptions the equations of magnetohydrodynamics [1] are written as

$$(\mathbf{u}\nabla)\,\mathbf{u} = -\frac{1}{M^2}\nabla p_1 + \beta(\mathbf{u}\times\mathbf{h})\times\mathbf{h},$$

$$\operatorname{div}\mathbf{u} = 0, \quad \Delta a = R_m \mathbf{u}\nabla a,$$

$$h_x = \frac{\partial a}{\partial y}, \quad h_y = -\frac{\partial a}{\partial x}$$

$$\left(\beta = R_m N^2, \quad N^2 = \frac{H_0^2}{4\pi\rho V_0^2}, \quad M^2 = \frac{\rho V_0^2}{P_0},\right.$$

$$\left. R_m = \frac{V_0 r_0}{\nu_m}, \quad \nu_m = \frac{c^2}{4\pi\sigma}\right). \qquad (1.1)$$

Here $\mathbf{h}$ is the magnetic field intensity normalized with respect to the magnetic field characteristic parameter H_0 inside the cylinder, $\mathbf{u}$ is the fluid velocity normalized with respect to the stream velocity V_0 as $x \to -\infty$, p_1 is the fluid pressure normalized with respect to the fluid pressure p_0 as $x \to -\infty$, and c is the speed of light. The x- and y-coordinates are normalized with respect to the cylinder characteristic dimension r_0, which for a circular cylinder is its radius.

We assume that the magnetic Reynolds number $R_m \gg 1$, and limit our analysis to flow patterns in which the perturbations of hydrodynamic parameters, induced by interaction with the magnetic field, are small. Then, we can linearize Eqs. (1.1) by letting $u = u_0 + v$, and $p_1 = p_{10} + p$, where p and v are small corrections to the dimensionless unperturbed stream pressure and velocity, i.e., of a stream flowing past a cylinder in which there is no current. It will be shown (Eqs. (3.3)) that the linear approximation holds for $N^2 \ll 1$. Linearizing (1.1), we obtain

$$-\mathbf{u}_0\times\operatorname{curl}\mathbf{v} = -\nabla\Phi_0 + \beta(\mathbf{u}_0\times\mathbf{h})\times\mathbf{h},$$

$$\Phi_0 = \mathbf{u}_0\cdot\mathbf{v} + p/M^2, \ \operatorname{div}\mathbf{v} = 0, \ \Delta a = R_m\mathbf{u}_0\cdot\nabla a. \qquad (1.2)$$

A similar problem was previously considered in [2]. However, the expression derived there for the magnetic-field vector potential does not hold in the neighborhood of the downstream critical point and moreover, the velocity field of the stream and the pressure distribution were erroneously calculated.

§ 2. MAGNETIC FIELD DISTRIBUTION

The magnetic field in the stream is defined by the last of Eqs. (1.2), and that within the cylinder by

$$\Delta a = -i^0. \qquad (2.1)$$

Here i^0 is the given current density distribution across the cylinder normalized with respect to $j_0 = cH_0/2\pi r_0$. Because of the magnetic field continuity along the cylinder surface the following conditions must be satisfied:

$$a_1 = a_2, \quad \partial a_1/\partial n = \partial a_2/\partial n, \qquad (2.2)$$

where n is the unit vector of the outward normal to the cylinder surface, and a_1 and a_2 are the vector potentials within the cylinder and in the stream, respectively. We look for a_1 and a_2 in the form of series expansions in powers of $1/R_m^{1/2}$

$$a_1 = a_{10} + \frac{1}{\sqrt{R_m}}a_{11} + \frac{1}{R_m}a_{12} + \ldots,$$

$$a_2 = a_{20} + \frac{1}{\sqrt{R_m}}a_{21} + \frac{1}{R_m}a_{22} + \ldots. \qquad (2.3)$$

Functions a_{1i} and a_{2i} (i = 0, ...) satisfy equations

$$\Delta a_{2i} = R_m\mathbf{u}_0\cdot\nabla a_{2i}, \qquad \Delta a_{1i} = -\delta_{i0}i^0, \qquad (2.4)$$

in which δ_{i0} is the Kronecker delta.

Noting that for $R_m \gg 1$ a magnetic boundary layer is formed in the stream surrounding the cylinder [3] in which $a_2 \sim 1/R_m^{1/2}$, while within the latter $a_1 \sim 1$ and $\partial/\partial n \sim 1$, we substitute Eqs. (2.3) into conditions (2.2) and equate terms of the same order with respect to $1/R_m^{1/2}$. From this we obtain the boundary conditions satisfied by a_{1i} and a_{2i} (i = 0, 1, 2, ...) at the cylinder surface

$$a_{10} = 0, \quad a_{11} = a_{20}, \quad a_{12} = a_{21}, \ \ldots, \ \partial a_{20}/\partial n = \partial a_{10}/\partial n,$$

$$\partial a_{21}/\partial n = \partial a_{11}/\partial n, \ldots.$$

If the expansions of a_1 and a_2 are restricted to their first terms, the problem of obtaining the magnetic field in the stream and in the cylinder can be divided into two parts. First, the case of the inner magnetic field defined by Eq. (2.1) with boundary condition $a_1 = 0$ is solved; then the external field problem, i.e., the fourth of Eqs. (1.2) with the following boundary conditions at the cylinder surface

$$\partial a_2/\partial n = \partial a_1/\partial n \qquad (2.6)$$

is solved.

Let us assume that the inner magnetic field is known and let us proceed to solve the external field problem. The potential function ξ and the stream function η of the unperturbed fluid flow are known, i.e.,

$$u_{0x} = \partial\xi/\partial x = \partial\eta/\partial y, \qquad u_{0y} = \partial\xi/\partial y = -\partial\eta/\partial x.$$

The arbitrary constant in the expression of ξ is selected so that the distance between the critical points equals unity. Passing in the last of Eqs. (1.2) to the new variables ξ and η, we obtain for a the following equation with constant coefficients

$$\partial^2 a/\partial\xi^2 + \partial^2 a/\partial\eta^2 = R_m \partial a/\partial\xi. \qquad (2.7)$$

Let the body and the current distribution across its section be symmetric with respect to the x-axis. We then have $u_0 \cdot h = 0$ outside of the body for $y = 0$ ($\eta = 0$). Since the inner magnetic field problem is assumed solved, and since the magnitude $u_0 \cdot h = u_0^2\, \partial/\partial\eta$ at the cylinder surface is known, for a in the stream we have the following boundary conditions:

$$\partial a/\partial\eta|_{\eta=0} = f(\xi), \nabla a \to 0 \text{ for } \sqrt{\xi^2+\eta^2} \to \infty,$$

$$f(\xi) = \mathbf{u}_0 \cdot \mathbf{h}/u_0^2|_{\eta=0}$$

$$\text{for } (0 < \xi < 1),\ f(\xi) = 0 \text{ for } (\xi < 0;\ \xi > 1).$$

Using these boundary conditions, we obtain a solution of (2.7), i.e.,

$$a = -\frac{1}{\pi}\int_0^1 \exp\left(\frac{R_m(\xi-\xi')}{2}\right) \times$$

$$\times K_0\left(\frac{R_m}{2}\sqrt{\eta^2+(\xi-\xi')^2}\right) f(\xi')\, d\xi'. \qquad (2.8)$$

Where K_0 is the MacDonald function.

Since $R_m \gg 1$, we find the asymptotic expression of a from (2.8) under the following conditions

$$R_m|\eta| \gg 1 \quad (0<\xi<1),\ R_m\sqrt{\eta^2+\xi^2} \gg 1 \quad (\xi<0),$$

$$R_m\sqrt{\eta^2+(1-\xi)^2} \gg 1 \qquad (\xi > 1). \qquad (2.9)$$

These conditions imply that a sufficiently narrow region adjoining the cylinder boundary ($\sim 1/R_m^{1/2}$) is excluded from our analysis. With conditions (2.9) satisfied for any ξ', we have $R_m(\eta^2 + (\xi - \xi)^2 \gg$ $\gg 1$, and K_0 can be replaced by its asymptotic expression. We further note that in the flow past the cylinder there are at its surface two critical points, i.e., the upstream and the downstream critical points at which $u = 0$. In the neighborhood of the upstream critical point $u_{0y} \sim \xi^{1/2}$, and consequently $f(\xi) \sim 1/\xi^{1/2}$, while in the vicinity of the downstream point $f(\xi) \sim 1/(1-\xi)^{1/2}$. We may therefore assume $f(\xi) = g(\xi)/\sqrt{\xi(1-\xi)}$, where the function $g(\xi)$ is regular along the segment (0.1). Considering this we derive from (2.8)

$$a = -\frac{2}{\sqrt{\pi R_m}}\exp\left(\frac{R_m\xi}{2}\right) \times$$

$$\times \int_0^{1/2\pi} \frac{\exp\{-{}^1/_2 R_m[\sin^2 u + \sqrt{\eta^2+(\xi-\sin^2 u)^2}]\}}{[\eta^2+(\xi-\sin^2 u)^2]^{1/4}} \times$$

$$\times g(\sin^2 u)\, du.$$

From (2.10) we obtain to within terms of order $\sim 1/R_m^{1/2}$

$$a \approx \frac{2}{R_m} g(0) \frac{\exp[-R_m r_1(\sin {}^1/_2\theta_1)^2]}{\sqrt{r_1}\sin {}^1/_2\theta_1}$$

$$\text{for } \sqrt{R_m r_1}\sin {}^1/_2\theta_1 \gg 1, \qquad (2.11)$$

$$\xi = r_1\cos\theta_1, \qquad \eta = r_1\sin\theta_1. \qquad (2.12)$$

We consider the flow past a cylinder of circular cross section with radius r_0 having a current-carrying conductor at its axis. In this case

$$\xi = {}^1/_2 + {}^1/_4(r + r^{-1})\cos\theta,$$

$$\eta = {}^1/_4(r - r^{-1})\sin\theta, \qquad g = -1.$$

In the neighborhood of the cylinder surface we assume $r = 1 +$ $+ t$ ($t \ll 1$). From (2.11) we obtain the asymptotic expression of a in the form

$$a = \frac{1}{R_m}\frac{\exp[-R_m t^2(\sin {}^1/_2\theta)^2]}{t\sin {}^1/_2\theta} \qquad \left(\sqrt{R_m}\, t\sin {}^1/_2\theta \gg 1\right). \qquad (2.13)$$

For this case a was calculated from Formula (2.8) with the aid of a computer. The results are shown on Figs. 2-4. The variation of h_θ with increasing radial distance from the cylinder surface is shown on Fig. 2 for several values of θ, and for $R_m = 2 \cdot 10^2$. The magnetic field lines plotted on Figs. 3 and 4 show the progressive elongation of these with increasing R_m in the form of a wake downstream of

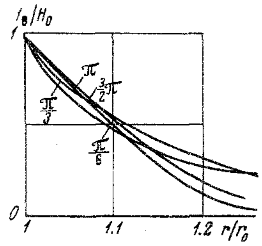

Fig. 2. Variation of the magnetic field with radial distance in the flow past a circular cylinder for various θ, and for $R_m = 2 \cdot 10^2$.

the cylinder; however, the field intensity in that region is low: $h \sim$ $\sim 1/R_m^{1/2}$. This can be readily demonstrated by assuming $\eta = 0$ in (2.10). The computations of a have also shown that for $r = 1$, $a(1, \theta) \sim$ $\sim 1/R_m^{1/2}$ everywhere, including the downstream critial point; hence expansions (2.3) are valid for any r and θ.

§3. VELOCITY FIELD PERTURBATIONS

We pass now to the investigation of the velocity field, and we introduce the stream function ψ. Then

$$v_x = \partial\psi/\partial y, \qquad v_y = -\partial\psi/\partial x$$

and for ψ we have from the first of Eqs. (1.2)

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = \frac{1}{u_0^2}\frac{\partial\Phi_0}{\partial\eta} + \beta\frac{\partial a}{\partial\xi}\frac{\partial a}{\partial\eta}$$

$$\left(\Phi_0 = -\beta\int_{-\infty}^{\xi} u_0^2\left(\frac{\partial a}{\partial\xi}\right)^2 d\xi\right). \qquad (3.1)$$

We begin with the analysis of the cylinder boundary layer.

a) Flow in the boundary layer. For $R_m \gg 1$ we have within the boundary layer $(\Delta\eta/\Delta\xi)^2 \sim 1/R_m \ll 1$, and the first term on the left-hand side of (3.1) can be neglected. Hence

$$\frac{\partial^2\psi}{\partial\eta^2} = \frac{1}{u_0^2}\frac{\partial\Phi_0}{\partial\eta} + \beta\frac{\partial a}{\partial\xi}\frac{\partial a}{\partial\eta}. \qquad (3.2)$$

Neglecting terms on the order of $\sim 1/R_m$, we introduce $1/u_0^2$ for $\partial/\partial\eta$ in (3.2), and using the fourth of Eqs. (1.2) we integrate (3.2), finally obtaining

$$\frac{\partial\psi}{\partial\eta} = \frac{\Phi_0}{u_0^2} + \frac{N^2}{2}\left(\frac{\partial a}{\partial\eta}\right)^2, \qquad \psi = \int_0^{\eta}\left[\frac{\Phi_0}{u_0^2} + \frac{N^2}{2}\left(\frac{\partial a}{\partial\eta}\right)^2\right]d\eta,$$

$$p = -\frac{N^2M^2}{2}h^2. \qquad (3.3)$$

b) Flow in the neighborhood of the downstream critical point. Equations (3.3) derived above do not hold in this region. Because of this we apply the curl operator to the first of Eqs. (1.2), integrate the obtained equation along the streamlines, and find

$$\mathrm{curl}_z \mathbf{v} = F \quad \left(F = \beta\int_{-\infty} \frac{1}{u_0}(\mathbf{h}\nabla)(\mathbf{u}_0\times\mathbf{h})_z\, dl_v\right). \qquad (3.4)$$

In a sufficiently small neighborhood of the downstream critical point the cylinder surface may be assumed flat. We introduce a local coordinate system with its origin at the critical point and the x- and y-axes, respectively, normal and tangent to the cylinder surface. In this coordinate system the function F may be represented for $r = (x^2 + y^2)^{1/2} \to 0$ in the form

$$F = \frac{y}{|y|} F_0, \tag{3.5}$$

where F_0 is the value of F at the downstream critical point.

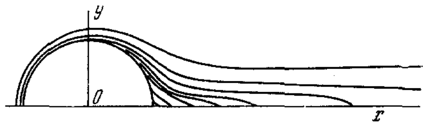

Fig. 3. Magnetic field lines in the flow past a current-carrying circular cylinder for $R_m = 2 \cdot 10^2$.

Thus, in the downstream critical-point neighborhood the function ψ must satisfy

$$\Delta\psi = \frac{y}{|y|} F_0 \tag{3.6}$$

and the boundary conditions

$$\psi|_{\theta=1/2\pi} = \psi|_{\theta=-1/2\pi} = 0. \tag{3.7}$$

Expanding the right-hand side of (3.6) and ψ into Fourier series and substituting into (3.6), we obtain

$$\psi = \frac{F_0}{\pi} r^2 \ln r \sin 2\theta - \frac{F_0}{\pi} r^2 \times$$

$$\times \sum_{k=1} \frac{\sin 2(2k+1)\theta}{(2k+1)[(k+1)^2-1]} + a_0 r^2 \sin 2\theta. \tag{3.8}$$

The last term of (3.8) relates to the unperturbed flow, and $a_0 \sim 1$. We will estimate F_0 for the case of flow past a circular cylinder with a current-carrying conduit at its axis. We write the explicit expression of F_0

$$F_0 = 2\beta \int_0^\pi [h_{0r}(\theta) \operatorname{ctg}\theta - h_{0r}^2(\theta)]\, d\theta \sim \sqrt{R_m} N^2$$

$$(h_{0r} = h_r|_{r=1}). \tag{3.9}$$

It follows from (3.8) that for any F_0 there exists a neighborhood of the downstream critical point in which the linear approximation does not hold (the additional term to the unperturbed stream function $\psi_0 = a_0 r^2 \sin 2\theta$ becomes ψ_0), and the dimension of this neighborhood for $R_m^{1/2} N^2 \gg 1$ is not less than $r^* \sim 1/R_m^{1/2}$ expression (3.6) is strictly applicable when $r \ll r^*$).

c) Velocity field distribution outside the boundary layer. The solution of Eq. (3.1) can be presented in the form

$$\psi = \frac{1}{4\pi} \int \ln [(\xi-\xi')^2 + (\eta-\eta')^2]\, \omega(\xi', \eta')\, d\xi'\, d\eta'$$

$$\left(\omega = \frac{1}{u_0^2} \frac{\partial \Phi_0}{\partial \eta} + \beta \frac{\partial a}{\partial \xi} \frac{\partial a}{\partial \eta}\right).$$

In the boundary layer around the cylinder and in the narrow downstream wake, the function ω differs in fact from zero, i.e., when $\eta \le \eta_0 \sim 1/R_m^{1/2}$. Taking this into account, and noting that $\omega(\xi', -\eta') = -\omega(\xi', \eta')$ for ψ when $\eta \gg \eta_0$ we obtain the expression

$$\psi = -\frac{\eta}{2\pi} \int \frac{g(\xi')\, d\xi'}{(\xi-\xi')^2 + \eta^2},$$

$$g(\xi') = I_1 + I_2,$$

$$I_1 = \int_{-\infty}^{\infty} \frac{\eta'}{u_0^2} \frac{\partial \Phi_0}{\partial \eta'} d\eta', \quad I_2 = \beta \int_{-\infty}^{\infty} \eta' \frac{\partial a}{\partial \xi'} \frac{\partial a}{\partial \eta'} d\eta'.$$

We will estimate integrals I_1 and I_2. Since u_0 remains virtually unchanged across the boundary layer,

$$I_1 \approx u_0^{-2}|_{\eta=0} \int_{-\infty}^{\infty} \eta' \frac{\partial \Phi_0}{\partial \eta'} d\eta' =$$

$$= -\frac{1}{u_0^2(\xi', 0)} \int_{-\infty}^{\infty} \Phi_0\, d\eta' \sim -\frac{\Phi_0(\xi', 0)}{u_0^2(\xi', 0)} \eta_0 \sim \frac{N^2}{\sqrt{R_m}} g_1(\xi').$$

In the boundary-layer approximation we have $\partial^2 a/\partial\eta^2 = R_m\, \partial a/\partial\xi$ from (2.7), hence

$$I_2 = \frac{N^2}{2} \int_{-\infty}^{\infty} \eta' \frac{\partial}{\partial \eta'} \left(\frac{\partial a}{\partial \eta'}\right)^2 d\eta' =$$

$$= -N^2 \int_0^\infty \left(\frac{\partial a}{\partial \eta'}\right)^2 d\eta' \sim -\frac{N^2}{\sqrt{R_m}} g_2(\xi').$$

Finally we obtain

$$g(\xi') = N^2 g_0(\xi') / \sqrt{R_m},$$

where $g_0(\xi)$ is a vector function independent of N and R_m. Consequently the perturbations of the velocity and pressure fields outside the boundary layer are on the order of $N^2/R_m^{1/2}$, and not of N^2, as derived in [2].

Let us discuss briefly the computation of the velocity field in [2]. Equation (3.1) can be written as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega_1(x, y) \quad \left(\omega_1 = -\frac{\partial \Phi_0}{\partial \eta} - \beta u_0^2 \frac{\partial a}{\partial \xi} \frac{\partial a}{\partial \eta}\right).$$

We now present ψ in the form

$$\psi = -\frac{1}{2\pi} \times$$

$$\times \int_0^\infty \int_{-\pi}^{\pi} \ln \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}\, \omega_1(r', \theta')\, r'\, dr'\, d\theta'.$$

Assuming that

$$\omega_1 = \omega_0(\theta)\, \delta(r-1) \quad \left(\omega_0 = \int_1^\infty \omega_1(r, \theta)\, dr\right)$$

(this is admissible because in the narrow boundary layer around the cylinder the function ω_1 differs from zero), from ψ we obtain the expression

$$\psi = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \sqrt{1 + r^2 - 2r \cos(\theta - \theta')}\, \omega_0(\theta')\, d\theta',$$

from which it is possible to derive

$$v_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{r - \cos(\theta - \theta')}{1 + r^2 - 2r \cos(\theta - \theta')} \omega_0(\theta')\, d\theta'.$$

If in the last expression we formally set $r = 1$, we obtain $v_\theta/_{r=1} = 0$, i.e., we obtain the result on which all computations of Section

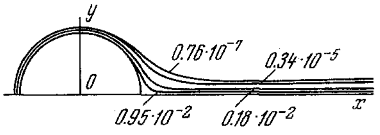

Fig. 4. Magnetic field lines in the flow past a current-carrying circular cylinder for $R_m = 2 \cdot 10^2$.

4 of [2] were based. This, however, is precisely what cannot be done, because ψ represents the potential of an ordinary layer, and $\partial\psi/\partial r$ is

discontinuous at r = 1. Hence

$$\lim_{r\to 1+0} \partial\psi / \partial r = {}^1\!/_2\omega_0(\theta), \text{ т. е. } v_\theta \Big|_{r=1} \neq 0.$$

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